SEMIDIRECT PRODUCT DECOMPOSITION OF COXETER GROUPS

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Abstract. Let (W,S) be a Coxeter system, let $S = I \cup J$ be a partition of S such that no element of I is conjugate to an element of J, let $\widetilde{J}$ be the set of W_I-conjugates of elements of J and let $\widetilde{W}$ be the subgroup of W generated by $\widetilde{J}$. We show that $W = \widetilde{W} \rtimes W_I$ and that $\widetilde{J}$ is the canonical set of Coxeter generators of the reflection subgroup $\widetilde{W}$ of W. We also provide algebraic and geometric conditions for an external semidirect product of Coxeter groups to arise in this way, and explicitly describe all such decompositions of (irreducible) finite Coxeter groups and affine Weyl groups.

Introduction

Let (W,S) be a Coxeter system and assume that S is the union of two subsets I and J such that no element of I is conjugate to an element of J. Let W_I be the subgroup of W generated by I. Let $\widetilde{J}$ be the set of elements of the form wsw^{-1} where w is in W_I and s is in J. Let $\widetilde{W}$ be the subgroup of W generated by $\widetilde{J}$. In [7], the following is shown:

Theorem (Gal). *With the above notation, we have:*

(a) $W = \widetilde{W} \rtimes W_I$ *(semidirect product with* $\widetilde{W}$ *normal).*

(b) $(\widetilde{W}, \widetilde{J})$ *is a Coxeter system*

Notation, remark, definition - Let $T = \bigcup_{w \in W} wSw^{-1}$ be the set of reflections of W. If $w \in W$, we set $N(w) = \{t \in T \mid \ell(wt) < \ell(w)\}$ where ℓ is the length function of (W,S). If W' is a subgroup of W generated by reflections, we set

$$\chi(W') = \{t \in T \mid N(t) \cap W' = \{t\}\}.$$

Then [6, (3.3)] $(W', \chi(W'))$ is a Coxeter system: $\chi(W')$ is called the set of *canonical Coxeter generators* of W'.

The following theorem clarifies the relation between the two natural sets $\widetilde{J}$, $\chi(\widetilde{W})$ of Coxeter generators of the normal reflection subgroup $\widetilde{W}$ of W.

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Theorem. *In the above setting,*

(a) $\widetilde{J} = \chi(\widetilde{W})$ *is the set of canonical Coxeter generators of* $\widetilde{W}$*.*

(b) *Each element* w *of* W_I *is the unique element of minimal length in its coset* $\widetilde{W}w = w\widetilde{W}$*.*

We remark that our result that $\widetilde{J} = \chi(\widetilde{W})$ is significantly stronger than the result of [7] that $(\widetilde{W}, \widetilde{J})$ is a Coxeter system, as it enables one to apply the favorable combinatorial and algebraic properties of the canonical Coxeter generators of reflection subgroups (see for example Lemma 1.2 and the remark after Corollary 1.3).

In this paper, we provide a simple algebraic proof of both theorems above, independent of the results of [7], and also describe algebraic conditions (Theorem 2.1) under which an external semidirect product of Coxeter groups is naturally a Coxeter group. We also provide an alternative proof (using root systems, see the proof of Theorem 3.6) of the fact that $\widetilde{J} = \chi(\widetilde{W})$ and of a formula for the Coxeter matrix of $(\widetilde{W}, \widetilde{J})$ obtained in [7]. Another main result is Theorem 3.11, which is a variant of Theorem 2.1, providing geometric conditions for the external semidirect product of two Coxeter systems to be a Coxeter system, when each is attached to a root system in the same ambient real vector space and the Coxeter group attached to the first root system acts as a group of automorphisms of the second based root system. We include some general results which are specific to the case in which W is finite or affine, including a construction of a homomorphism between Solomon descent algebras of W and $\widetilde{W}$ when W is finite. Finally, we describe explicitly by tables the internal semidirect product decompositions (as above) of (irreducible) finite Coxeter groups and affine Weyl groups.

COMMENT - Semidirect product decompositions as above are used by the first author [2] in studying the Hecke algebra and the Kazhdan-Lusztig theory with unequal parameters whenever the parameters are zero on the set I. In fact, the authors were unaware of [7] and originally obtained both theorems above, together with the same description of the Coxeter matrix of $(\widetilde{W}, \widetilde{J})$ as in [7], by the above-mentioned root-system arguments. We are indebted to Koji Nuida for bringing [7] to our attention.

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1. Internal semidirect products

For convenience, we restate the two Theorems of the introduction here in combined form.

Theorem 1.1. *With the notation of the introduction, we have:*

(a) $W = \widetilde{W} \rtimes W_I$ *(semidirect product with $\widetilde{W}$ normal).*
(b) $(\widetilde{W}, \widetilde{J})$ *is a Coxeter system.*
(c) $\widetilde{J} = \chi(\widetilde{W})$ *is the set of canonical Coxeter generators of $\widetilde{W}$.*
(d) *Each element w of W_I is the unique element of minimal length in its coset $\widetilde{W}w = w\widetilde{W}$.*

Proof. If s and t are elements of T, we denote by $m_{s,t}$ the order of st. It is well known that two simple reflections are W-conjugate iff, regarded as vertices of the Coxeter graph of (W, S), there is a path from one to the other such that each edge of the path has either an odd label or no label (i.e. a label of 3, which is omitted by the standard convention). In particular:

$$\text{(1.1)} \qquad \textit{If } s \in I \textit{ and } t \in J, \textit{ then } m_{s,t} \textit{ is even.}$$

We first prove (a). Let

$$\begin{array}{rccl} \varphi : & S & \longrightarrow & W_I \\ & s & \longmapsto & \begin{cases} s & \text{if } s \in I, \\ 1 & \text{otherwise.} \end{cases} \end{array}$$

It follows easily from (1.1) that $(\varphi(s)\varphi(t))^{m_{st}} = 1$ for all s, $t \in S$. Therefore, there exists a unique morphism of groups $W \to W_I$ extending φ: we still denote it by φ. Since $\varphi(w) = w$ for all $w \in W_I$, it is sufficient to prove that

$$\text{(1.2)} \qquad \operatorname{Ker} \varphi = \widetilde{W}.$$

Let us prove (1.2). First of all, note that $\widetilde{W} \subseteq \operatorname{Ker} \varphi$. So it is enough to show that $W = \widetilde{W} W_I$. For this, it is sufficient to show that if $w \in W \setminus W_I$, there is some $t \in \widetilde{J}$ with $l(tw) < l(w)$. Write $w = s_1 \cdots s_n$ (reduced) with all $s_i \in S$. Since $w \notin W_I$, there is some j with $s_j \in J$. Without loss of generality, assume that j is minimal with this property. Then $t := s_1 \cdots s_{j-1} s_j s_{j-1} \cdots s_1 \in \widetilde{J}$ and $l(tw) < l(w)$ as required. This completes the proof of (a).

We claim next that $\widetilde{W} \cap T$ consists of all W-conjugates of elements of J. In fact, since $\widetilde{W}$ is generated by $\widetilde{J} \subseteq T$, [6, 3.11(ii)] implies that $\widetilde{W} \cap T$ consists of the $\widetilde{W}$-conjugates of elements of $\widetilde{J}$; since $\widetilde{J}$ consists of the W_I-conjugates of elements of J, (a) implies that the $\widetilde{W}$-conjugates of elements of $\widetilde{J}$ are exactly the W-conjugates of elements of J, completing the proof of the claim.

We can now prove (c) (which immediately implies (b)). Regard the power set $\mathscr{P}(T)$ of T as an abelian group under symmetric difference $A + B := (A \cup B) \setminus (A \cap B)$

and with natural W-action $(w, A) \mapsto wAw^{-1} := \{\, wtw^{-1} \mid t \in A \,\}$. The function $N\colon W \to \mathscr{P}(T)$ is characterized by the cocycle condition $N(xy) = N(x) + xN(y)x^{-1}$ for $x, y \in W$, and its special values $N(s) = \{s\}$ for $s \in S$ (see [6]). Consider an element $t \in \widetilde{J}$, say $t = wrw^{-1}$ where $w \in W_I$ and $r \in J$. We have

$$N(t) = N(wrw^{-1}) = N(w) + wN(r)w^{-1} + wrN(w^{-1})rw^{-1}$$

Note that for $x \in W_I$, $N(x) \subseteq W_I \cap T$ consists of reflections which are W-conjugate to elements of I, and hence all elements of $N(w) \cup wrN(w^{-1})rw^{-1}$ are W-conjugate to elements of I. From the claim and the assumption that no element of I is W-conjugate to an element of J, it therefore follows that $N(t) \cap \widetilde{W} = wN(r)w^{-1} \cap \widetilde{W} = \{t\}$ i.e. $t \in \chi(\widetilde{W})$. This proves that $\widetilde{J} \subseteq \chi(\widetilde{W})$. Now $\chi(\widetilde{W})$ is a set of Coxeter generators of $\widetilde{W}$, and hence it is a minimal (under inclusion) set of generators of $\widetilde{W}$ by [4, Ch IV, §1, Cor 3]. Since $\widetilde{J}$ generates $\widetilde{W}$ by definition, we get (b)–(c)

We now state a Lemma needed in the proof of (d) (and elsewhere in this paper).

Lemma 1.2. *Let W' be a subgroup of W generated by reflections, let $S' = \chi(W')$ and let X' be the set of elements $x \in W$ such that x has minimal length in xW'. Then:*

Every coset in W/W' contains a unique element of X'.

(b) *An element $x \in W$ belongs to X' if and only if $\ell(xt) > \ell(x)$ for all $t \in S'$.*

(c) *If $x \in X'$, $w \in W'$ and $t \in W' \cap T$, then $\ell(xwt) > \ell(xw)$ iff $\ell(wt) > \ell(w)$ iff $\ell'(wt) > \ell'(wt)$ where ℓ' is the length function of (W', S').*

Proof. See [6, (3.4)]. □

If $t \in \widetilde{J}$ and $w \in W_I$, then $l(wt) > l(w)$ since $t \notin W_I$. Since $\widetilde{J}$ is the set of canonical generators of $\widetilde{W}$, this implies that $w \in W_I$ is the (unique) element of minimal length in its coset $w\widetilde{W}$ by Lemma 1.2 (a)–(b). Theorem 1.1 (d) follows from the fact that $W = W_I\widetilde{W}$. □

In the remainder of this section, we give some simple complements to and consequences of Theorem 1.1, with applications in [2] or to explicit computation in examples, and then describe the Coxeter matrix of $(\widetilde{W}, \widetilde{J})$.

Let $\tilde{\ell}\colon \widetilde{W} \to \mathbb{N}$ denote the length function of $(\widetilde{W}, \widetilde{J})$. If $w \in W$, we denote by $\ell_I(w)$ (respectively $\ell_J(w)$) the number of occurrences of elements of I (respectively J) in a reduced expression of w (note that these two numbers do not depend on the choice of the reduced expression and that $\ell(w) = \ell_I(w) + \ell_J(w)$).

Corollary 1.3. *Let $a \in W_I$ and $w \in \widetilde{W}$. Then $\ell_J(aw) = \ell_J(wa) = \ell_J(w) = \tilde{\ell}(w)$ and $\tilde{\ell}(awa^{-1}) = \tilde{\ell}(w)$.*

Proof. Since W_I acts on $\widetilde{W}$ by preserving $\widetilde{J}$, the length function $\tilde{\ell}$ is invariant by W_I-conjugation, so $\tilde{\ell}(awa^{-1}) = \tilde{\ell}(w)$. Also, if $s \in I$ and $x \in W$, we have $\ell_J(sx) = \ell_J(x) = \ell_J(xs)$, so this shows that $\ell_J(aw) = \ell_J(wa) = \ell_J(w)$.

It remains to show that $\ell_J(w) = \tilde{\ell}(w)$. We argue by induction on $\tilde{\ell}(w)$. The result is clear if $\tilde{\ell}(w) = 0$. So assume that $\tilde{\ell}(w) > 0$. Then there exists $\tilde{t} \in \widetilde{J}$ such that $\tilde{\ell}(\tilde{t}w) < \tilde{\ell}(w)$. Let $x \in W_I$ and $t \in J$ be such that $\tilde{t} = xtx^{-1}$. Let $w' = x^{-1}wx$. Since $\tilde{\ell}$ is invariant by W_I-conjugation, we get that $\tilde{\ell}(w) = \tilde{\ell}(w')$ and $\tilde{\ell}(\tilde{t}w) = \tilde{\ell}(tw')$. Therefore $\tilde{\ell}(tw') = \tilde{\ell}(w') - 1$ and so, by Lemma 1.2 (c), we get that $\ell(tw') = \ell(w') - 1$. In particular, $\ell_J(tw') = \ell_J(w') - 1$. So, by the induction hypothesis, $\ell_J(w') - 1 = \ell_J(tw') = \tilde{\ell}(tw') = \tilde{\ell}(w') - 1$. Therefore, $\ell_J(w') = \tilde{\ell}(w')$ and so $\ell_J(w) = \ell_J(w') = \tilde{\ell}(w') = \tilde{\ell}(w)$. ☐

Remark - We observe that the Corollary is not an obvious consequence of the results proved in [7]; for example, the proof above requires Theorem 1.1(c), and not just Theorem 1.1(a)–(b).

The next results require some additional notation. If $\tilde{t} \in \widetilde{J}$ and if t and $t' \in J$ and x, $x' \in W_I$ are such that $\tilde{t} = xtx^{-1} = x't'x'^{-1}$, then

$$t = t'. \tag{1.3}$$

Indeed, in this case, then $t' \in W_{I \cup \{t\}} \cap W_J =< t >$. Therefore, if $\tilde{t} \in \widetilde{J}$, we can define $\nu(\tilde{t})$ as the unique element of J which is conjugate to $\tilde{t}$ under W_I.

Corollary 1.4. *If $\tilde{t}$, $\tilde{t}' \in \widetilde{J}$ are $\widetilde{W}$-conjugate, then $\nu(\tilde{t})$ and $\nu(\tilde{t}')$ are W-conjugate.*

Remark - Recall that an isomorphism of Coxeter systems $(W_1, S_1) \to (W_2, S_2)$ is a group isomorphism $W_1 \to W_2$ inducing a bijection $S_1 \to S_2$. In the semidirect product decomposition $W = \widetilde{W} \rtimes W_I$ of Theorem 1, it is clear that the induced action by conjugation of W_I on $\widetilde{W}$ is by automorphisms of the Coxeter system $(\widetilde{W}, \widetilde{J})$. Moreover, the set of Coxeter generators S of W is the disjoint union of the set I of Coxeter generators of W_I and the set J of W_I-orbit representatives on $\widetilde{J}$.

In order to parametrize $\widetilde{J}$, we must first determine the centralizer of $t \in J$ in W_I. If $s \in S$, we set $s^\perp = \{r \in S \mid sr = rs\}$.

Lemma 1.5. *Let $t \in J$. Then $C_{W_I}(t) = W_{I \cap t^\perp}$.*

Proof. First it is clear that $W_{I \cap t^\perp} \subseteq C_{W_I}(t)$. Conversely, let $w \in W_I$ be such that $wt = tw$. Let $w = s_1 \cdots s_r$ be a reduced expression of w (so that $s_i \in I$). Then, $s_1 \cdots s_r t$ and $ts_1 \cdots s_r$ are reduced expression of the same element $wt = tw$ of W. By Matsumoto's lemma, this means that one can obtain one of these reduced expression by applying only braid relations. But t occurs only once in both reduced expressions:

this means that, in order to make t pass from the first position to the last position, t must commute with all the s_i. So $w \in W_{I\cap t^\perp}$. $\square$

We set

$$\mathscr{J} = \{(x,t) \mid t \in J \text{ and } x \in X^I_{I\cap t^\perp}\}.$$

Then it follows from (1.3) and Lemma 1.5 that the map

$$j : \begin{array}{ccc} \mathscr{J} & \longrightarrow & \widetilde{J} \\ (x,t) & \longmapsto & xtx^{-1} \end{array} \tag{1.4}$$

is bijective.

Proposition 1.6. *Let $(x,t) \in \mathscr{J}$. Then:*

(a) *For $w \in W_I$, one has $wj(x,t)w^{-1} = j(x',t)$ where x' is the unique element of $X^I_{I\cap t^\perp}$ with $x'W_{I\cap t^\perp} = wxW_{I\cap t^\perp}$.*

(b) *The palindromic reduced expressions of xtx^{-1} in (W,S) are precisely the expressions $t_n \cdots t_1 t_0 t_1 \cdots t_n$ such that $t_n \cdots t_1$ is a reduced expression for x in (W_I, I) and $t_0 = t$.*

Proof. Part (a) is immediate from the definitions. For (b), we first recall the following result:

> **Lemma 1.7.** *If $r_1 \cdots r_{2m+1}$ is a reduced expression for a reflection $t \in T$, then $r_1 \cdots r_m r_{m+1} r_m \cdots r_1$ is a palindromic reduced expression of t.*
>
> *Proof.* See [6, (2.7)]. $\square$

Write $l(xtx^{-1}) = 2m+1$. We have $xtx^{-1} \in W_{I\cup\{t\}}$, so any reduced expression $xtx^{-1} = s_1 \cdots s_{2m+1}$ for xtx^{-1} has all $s_i \in I \cup \{t\}$. Note $s_1 \cdots s_m s_{m+1} s_m \cdots s_1$ is also a reduced expression for xtx^{-1} by Lemma 1.7. Thus, $t \in J$ is W-conjugate to $s_{m+1} \in I \cup \{t\}$ and so $s_{m+1} = t$. Let $t_n \cdots t_1$ be a reduced expression for x, and $t_0 = t$. Then $xtx^{-1} = t_n \cdots t_1 t_0 t_1 \cdots t_n$ and the right hand side contains some reduced expression $s_1 \cdots s_{2m+1}$ for xtx^{-1} as a subexpression. By the above, we have $s_{m+1} = t = t_0$, which is the only occurrence of t in $t_n \cdots t_0 \cdots t_n$. Hence $s_1 \cdots s_m s_{m+1} s_m \cdots s_1$ is also a reduced expression for xtx^{-1} contained as a subexpression of $t_n \cdots t_0 \cdots t_1$. Let $y = s_1 \cdots s_m \in W_I$. Then $xtx^{-1} = yty^{-1}$ so $z := y^{-1}x \in C_{W_I}(t) = W_{I\cap t^\perp}$. We have $y = xz^{-1}$ with $\ell(y) = m = \ell(xz^{-1}) = \ell(x) + \ell(z^{-1}) = n + \ell(z^{-1})$ and $m \leqslant n$, so $m = n$. This shows $t_n \cdots t_0 \cdots t_n$ is a reduced expression for xtx^{-1}.

Since every reduced expression for xtx^{-1} has t as its middle element, it follows that this central t can never be involved in a braid move between reduced expressions for xtx^{-1}, and the conclusion of (b) is clear. $\square$

Now we introduce notation to describe the Coxeter matrix of $(\widetilde{W}, \widetilde{J})$. If A, B and C are three subsets of S such that $B \subseteq A$ and $C \subseteq A$, we denote by X_{BC}^A the set of $x \in W_A$ which have minimal length in $W_B x W_C$. For simplicity, we set $X_{\varnothing C}^A = X_C^A$. We shall use Deodhar's Lemma [8, Lemma 2.1.2], which amounts to to the statement that if $w \in W_C^A$ and $s \in A$ with $sw \notin W_C^A$ then $\ell(sw) > \ell(w)$ and $sw = wr$ for some $r \in C$.

Now, let $\tilde{s}$, $\tilde{t} \in \widetilde{J}$ and let $s = \nu(\tilde{s})$ and $t = \nu(\tilde{t})$. Then there exists x and $y \in W_I$ such that $\tilde{s} = xsx^{-1}$ and $\tilde{t} = yty^{-1}$. We denote by $f(\tilde{s}, \tilde{t})$ the unique element of $X^I_{I \cap s^\perp, I \cap t^\perp}$ such that $x^{-1}y \in W_{I\cap s^\perp} f(\tilde{s}, \tilde{t}) W_{I \cap t^\perp}$. It is readily seen that $f(\tilde{s}, \tilde{t})$ depends only on $\tilde{s}$ and $\tilde{t}$ and not on the choice of x and y. Note that $\tilde{s} = \tilde{t}$ if and only if $s = t$ and $f(\tilde{s}, \tilde{t}) = 1$. Recall that if s and t are two elements of T, $m_{s,t}$ denotes the order of st. We then set:

$$\tilde{m}_{\tilde{s},\tilde{t}} = \begin{cases} 1 & \text{if } \tilde{s} = \tilde{t}, \\ m_{s,u}/2 & \text{if } s = t \text{ and } f(\tilde{s}, \tilde{t}) = u \in I, \\ \infty & \text{if } s = t \text{ and } \ell(f(\tilde{s}, \tilde{t})) \geqslant 2, \\ m_{s,t} & \text{if } s \neq t \text{ and } f(\tilde{s}, \tilde{t}) = 1, \\ \infty & \text{if } s \neq t \text{ and } f(\tilde{s}, \tilde{t}) \neq 1. \end{cases}$$

We denote by $\widetilde{M}$ the matrix $(\tilde{m}_{\tilde{s},\tilde{t}})_{\tilde{s},\tilde{t} \in \widetilde{J}}$.

Since $f(\tilde{s}, \tilde{t}) = f(\tilde{t}, \tilde{s})^{-1}$ and since m_{su} is even if $s \in J$ and $u \in I$ by (1.1), we have, for all $\tilde{s}$, $\tilde{t} \in \widetilde{J}$ and $x \in W_I$,

$$\begin{cases} \tilde{m}_{\tilde{s},\tilde{t}} \in \mathbb{Z}_{\geqslant 1}, \\ \tilde{m}_{\tilde{s},\tilde{t}} = \tilde{m}_{\tilde{t},\tilde{s}}, \\ \tilde{m}_{x\tilde{s}x^{-1}, x\tilde{t}x^{-1}} = \tilde{m}_{\tilde{s},\tilde{t}}, \\ \tilde{m}_{\tilde{s},\tilde{s}} = 1, \\ \tilde{m}_{\tilde{s},\tilde{t}} \geqslant 2 \text{ (if } \tilde{s} \neq \tilde{t}). \end{cases} \tag{1.5}$$

The last inequality follows from the fact that, if $f(\tilde{s}, \tilde{t}) = u \in I$, then $us \neq su$.

The following is proved by a simple algebraic argument, independent of the proof of Theorem 1.1, in [7]. For a different proof using root systems, see Theorem 3.6.

Theorem 1.8 (Gal)**.** *For $\tilde{s}, \tilde{t} \in \widetilde{J}$, the product $\tilde{s}\tilde{t}$ in W has order $\tilde{m}_{\tilde{s},\tilde{t}}$ i.e. the matrix $\widetilde{M}$ defined above is the Coxeter matrix of $(\widetilde{W}, \widetilde{J})$.*

Corollary 1.9. *Let L be a subset of $\widetilde{J}$ such that the Coxeter graph of $(\widetilde{W}_L, L)$ is an irreducible component of the one of $(\widetilde{W}, \widetilde{J})$. Then $J = \{\nu(\tilde{t}) \mid \tilde{t} \in L\}$.*

Proof. Let $K = \{\nu(\tilde{t}) \mid \tilde{t} \in L\} \subseteq J$. Then K is not empty. We shall prove by induction on n the following assertion (from which the Corollary follows easily):

(P_n) *Let $s \in K$ and let $t \in J$ be such that there exists a path of length n from s to t in the Coxeter graph of (W, S). Then $t \in K$.*

It is clear that (P_0) holds. Let us show (P_1). So let $s \in K$ and $t \in J$ be such that $m_{s,t} \geqslant 3$. Then there exists $x \in W_I$ such that $xsx^{-1} \in L$. Since $m_{xsx^{-1},xtx^{-1}} = m_{s,t} \geqslant 3$ and $xtx^{-1} \in \widetilde{J}$, we get that $xtx^{-1} \in L$ and so $t = \nu(xtx^{-1}) \in K$, as expected.

Now let $n \geqslant 2$ and assume that (P_0), (P_1),..., (P_{n-1}) hold. Let $s \in K$ and let $t \in J$ be such that there exists a path of length n from s to t in the Coxeter graph of (W, S). Set $s_0 = s$ and $s_n = t$ and let s_1,..., s_{n-1} be elements of S such that $m_{s_{i-1},s_i} \geqslant 3$ for all $i \in \{1, 2, \dots, n\}$. We may assume that $s_i \neq s_j$ if $i \neq j$. If there exists $i \in \{1, 2, \dots, n-1\}$ such that $s_i \in J$, then the induction hypothesis (applied twice) implies that $s_i \in K$ and that $s_n = t \in K$. So we may assume that $s_i \in I$ for all $i \in \{1, 2, \dots, n-1\}$.

Let $x \in W_I$ be such that $\tilde{s} = xsx^{-1} \in L$. Let $y = s_1 s_2 \cdots s_{n-1} \in W_I$ and let $\tilde{t} = xyty^{-1}x^{-1}$. Note that $\ell(y) = n - 1$. Since $s \neq t$, we get that $m_{\tilde{s},\tilde{t}} = \infty$ if $y \notin W_{I \cap s^\perp} \cdot W_{I \cap t^\perp}$. So it remains to show that $y \notin W_{I \cap s^\perp} \cdot W_{I \cap t^\perp}$. But, if $y \in W_{I \cap s^\perp} \cdot W_{I \cap t^\perp}$, this would imply that y has a reduced expression of the form $y = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$ such that there exists $k \in \{0, 1, 2, \dots, n-1\}$ satisfying σ_1,..., $\sigma_k \in I \cap s^\perp$ and $\sigma_{k+1}, \dots, \sigma_{n-1} \in I \cap t^\perp$. Since y has only one reduced expression, this means that $s_1 \in I \cap s^\perp$ or $s_{n-1} \in I \cap t^\perp$. This is impossible, and so the proof of (P_n) is complete. □

Corollary 1.10. *Assume that (W, S) is irreducible. Then W_I permutes transitively the irreducible components of $(\widetilde{W}, \widetilde{J})$.*

Proof. Let L and L' be two subsets of $\widetilde{J}$ such that the Coxeter graphs of $(\widetilde{W}_L, L)$ and $(\widetilde{W}_{L'}, L')$ are irreducible components of $(\widetilde{W}, \widetilde{J})$. Let $s \in J$. By Corollary 1.9, there exist x and y in W_I such that $xsx^{-1} \in L$ and $ysy^{-1} \in L'$. So $L \cap {}^{xy^{-1}}L' \neq \varnothing$. Since W_I permutes the irreducible components of the Coxeter graph of $(\widetilde{W}, \widetilde{J})$, we get that $L = {}^{xy^{-1}}L'$. □

Parabolic subgroups, cosets. We close this section by investigating the relationships between standard parabolic subgroups of W and $\widetilde{W}$, as well as between the sets of distinguished cosets representatives. Roughly speaking, with respect to these questions, $\widetilde{W}$ behaves like a standard parabolic subgroup.

If L is a subset of $\widetilde{J}$, we note by $\widetilde{W}_L$ the subgroup of $\widetilde{W}$ generated by L. If K is a subset of S, we set

$$K^+ = \{wtw^{-1} \mid w \in W_{I \cap K} \text{ and } t \in J \cap K\}.$$

It is a subset of $\widetilde{J}$. Then

Proposition 1.11. *Let K be a subset of S. Then $K^+ = W_K \cap \widetilde{J}$ and $W_K \cap \widetilde{W} = \widetilde{W}_{K^+}$.*

Proof. Let $\varphi_K : W_K \to W$ denote the restriction to W_K of the morphism $\varphi : W \to W_I$ defined in the proof of Theorem 1.1. Then $\widetilde{W}_{K^+} \subseteq \operatorname{Ker}\varphi_K$ and, if $w \in W_{I\cap K}$, we have $\varphi_K(w) = w$. But, by Theorem 1.1 applied to W_K and to the partition $K = (I \cap K) \dot{\cup} (J \cap K)$, we have $W_K = W_{I\cap K} \ltimes \widetilde{W}_{K^+}$. Therefore, $\widetilde{W}_{K^+} = \operatorname{Ker}\varphi_K$. But, by 1.2, $\operatorname{Ker}\varphi_K = \widetilde{W} \cap W_K$. This shows the second equality of the proposition. The first one then follows easily. □

If L is a subset of $\widetilde{J}$, we denote by $\widetilde{X}_L$ (respectively X_L) the set of elements w of $\widetilde{W}$ (respectively W) which have minimal length in $w\widetilde{W}_L$.

Lemma 1.12. *Let L be a subset of $\widetilde{J}$. Then the map*

$$\begin{array}{ccc} W_I \times \widetilde{X}_L & \longrightarrow & X_L \\ (w,x) & \longmapsto & wx \end{array}$$

is bijective.

Proof. First, it follows from Theorem 1.1 (a) that the map

$$\begin{array}{ccc} W_I \times \widetilde{X}_L & \longrightarrow & W/\widetilde{W}_L \\ (w,x) & \longmapsto & wx\widetilde{W}_L \end{array}$$

is bijective. So it remains to show that, if $w \in W_I$ and $x \in \widetilde{X}_L$, then $wx \in X_L$. But this follows from Lemma 1.2 (c). □

We conclude by an easy result on double coset representatives:

Proposition 1.13. *Let K be a subset of S. Then the map*

$$\begin{array}{ccc} X_{I\cap K}^I & \longrightarrow & \widetilde{W}\backslash W/W_K \\ d & \longmapsto & \widetilde{W}dW_K \end{array}$$

is bijective. Moreover, if $d \in X_{I\cap K}^I$, then d is the unique element of minimal length in $\widetilde{W}dW_K = d\widetilde{W}W_K$ and

$$\widetilde{W} \cap {}^dW_K = \widetilde{W}_{\widetilde{J}\cap {}^dW_K}.$$

Proof. First, $\widetilde{W}\backslash W/W_K = W/\widetilde{W}W_K = W/W_K\widetilde{W}$ and, since $W_K = W_{I\cap K} \ltimes \widetilde{W}_{K^+}$, we have

$$W/W_K\widetilde{W} = W/(W_{I\cap K} \ltimes \widetilde{W}) \simeq W_I/W_{I\cap K}.$$

This shows the first assertion.

Now, let $d \in X_{I\cap K}^I$. Then, since $\widetilde{W}$ is normal in W, we get

$$\widetilde{W} \cap {}^dW_K = {}^d(\widetilde{W} \cap W_K) = {}^d\widetilde{W}_{K^+}$$

(see Proposition 1.11). But W_I acts on the pair $(\widetilde{W}, \widetilde{J})$, so

$$^{d}\widetilde{W}_{K^+} = \widetilde{W}_{^{d}K^+}.$$

Now, by Proposition 1.11, we have

$$^{d}K^+ = {}^{d}(\widetilde{J} \cap W_K) = \widetilde{J} \cap {}^{d}W_K.$$

So the last assertion follows.

It remains to show that d is the unique element of minimal length in $\widetilde{W} d W_K$. We have $\widetilde{W} d W_K = d(W_{I\cap K} \ltimes \widetilde{W})$. Let $x \in W_{I\cap K}$ and $w \in \widetilde{W}$ be such that $\ell(dxw) \leqslant \ell(d)$. Then, by Theorem 1.1 (d), $\ell(dxw) \geqslant \ell(dx) = \ell(d) + \ell(x)$, so $x = 1$. Again by Theorem 1.1 (d), we get $\ell(dw) = \ell(d)$, so $w = 1$, as expected. □

2. External semidirect products

In this section, we discuss the converse of Theorem 1.1(a)–(b), giving conditions which imply that an external semidirect product of Coxeter groups is a Coxeter group.

Let (W', I) and $(\widetilde{W}, \widetilde{J})$ be Coxeter systems and $\theta\colon W' \to \mathrm{Aut}(\widetilde{W}, \widetilde{J})$ be a group homomorphism, where the right hand side is the group of automorphisms of $(\widetilde{W}, \widetilde{J})$. One may regard θ as a homomorphism from W' to the automorphism group of $\widetilde{W}$, and form the semidirect product of groups $W := \widetilde{W} \rtimes W'$, with $\widetilde{W}$ normal. We regard W' and $\widetilde{W}$ as subgroups of W in the usual way. Thus, every element w of W has a unique expression $w = \widetilde{w}w'$ with $w' \in W'$ and $\widetilde{w} \in \widetilde{W}$. The product in W is determined by the equation $w'\widetilde{w}w'^{-1} = \theta(w')(\widetilde{w})$ for $w' \in W'$, $\widetilde{w} \in \widetilde{W}$.

Theorem 2.1. *Fix a set J of W_I-orbit representatives on $\widetilde{J}$, and set $S := I \cup J$. For any $s \in S$, let $s^\perp := \{\, r \in S \mid rs = sr \,\}$. Then (W, S) is a Coxeter system iff the conditions* (1) *and* (2) *below hold:*

(1) *for all $r, s \in J$ and $u \in W'$ with $r = usu^{-1}$, one has $r = s$ and $u \in W'_{I\cap r^\perp}$.*

(2) *for all $r \in J$ and $s \in \widetilde{J}$ with $r \neq s$ and rs of finite order, either* (i) *or* (ii) *below holds:*

  (i) *$s = utu^{-1}$ for some $u \in W'_{I\cap r^\perp}$ and $t \in J$ with $t \neq r$ and rt of finite order*

  (ii) *$s = uvrvu^{-1}$ for some $u \in W'_{I\cap r^\perp}$ and $v \in I$ with rv of finite order greater than 2.*

Proof. It is easy to see that S is a set of involutions generating W. No element of I is W-conjugate to an element of J (since any W-conjugate of an element of J is in $\widetilde{W}$); in particular, the union $S = I \dot\cup J$ is disjoint (we shall use $\dot\cup$ to denote disjoint union throughout this paper). Moreover, a simple computation shows that for $s \in I$ and $r \in J$, the order of sr in W is even, equal to twice the order of $r'r$ in $\widetilde{W}$ where $r' = \theta(s)(r) = srs$.

For $r, s \in S$, let $m_{r,s}$ denote the order of rs. We have $m_{r,r} = 1$ and $m_{r,s} = m_{s,r} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for all $r \neq s$. Let (U, S) be a Coxeter system with Coxeter matrix $m_{r,s}$ i.e. U is a Coxeter group with S as its set of Coxeter generators, and the order of rs in U is $m_{r,s}$ for all $r, s \in S$.

For any $K \subseteq S$, let U_K denote the standard parabolic subgroup of U generated by K. Let $\widetilde{J}'$ denote the subset of U consisting of all products usu^{-1} in U with $s \in J$ and $u \in U_I$, and let $\widetilde{U}$ denote the subgroup of U generated by $\widetilde{J}'$. No element of I is conjugate in U to an element of J, since $m_{r,s}$ is even for all $r \in I$ and $s \in J$. Hence, by Theorem 1.1, there is a semidirect product decomposition $U = U_I \ltimes \widetilde{U}$ with $\widetilde{U}$ normal in U.

Since rs has the same order $m_{r,s}$ in both U and W, for any $r, s \in S$, there is a group epimorphism $\pi\colon U \to W$ which is the identity on S. The homomorphism π restricts to an isomorphism of Coxeter systems $(U_I, I) \to (W', I)$ (which we henceforward regard as an identification) and π also restricts to an isomorphism of Coxeter systems $(U_J, J) \to (\widetilde{W}_J, J)$. Further, π restricts to a surjective, W'-equivariant (for the conjugation actions by W') group homomorphism $\tilde{\pi}\colon \widetilde{U} \to \widetilde{W}$ and $\tilde{\pi}$ restricts further to a surjective map of W'-sets $\pi'\colon \widetilde{J}' \to \widetilde{J}$.

Now if (W, S) is a Coxeter system, the validity of the conditions of Theorem 2.1 (1)–(2) follows readily from (1.4) and Theorem 1.8. (In this case, the map $\tilde{\pi}$ is of course an isomorphism of Coxeter systems).

Conversely, suppose that (1) and (2) hold. It will suffice to show that $\tilde{\pi}$ is an isomorphism of Coxeter systems. First, we show that π' is injective. Consider two arbitrary elements uru^{-1} and vsv^{-1} of $\widetilde{J}'$, with $u, v \in W'$ and $r, s \in J$. Assume $\pi(uru^{-1}) = \pi(vsv^{-1})$ i.e. $u\pi(r)u^{-1} = v\pi(s)v^{-1}$. Then $\pi(r) = x\pi(s)x^{-1}$ where $x = u^{-1}v \in W'$. By (1), $r = s$ and $x \in W'_{I\cap r^\perp}$. By the defining relations for (U, S), it follows that $r = xsx^{-1}$ in U, so $uru^{-1} = vsv^{-1}$ in U. Hence π' is injective, and in fact bijective since we noted above that π' is a surjection.

Now it will suffice to show that for all distinct $r', s' \in \widetilde{J}'$, $r's'$ has the same order in U as $\pi(r')\pi(s')$ has in W. Using the W'-equivariance of $\tilde{\pi}$, we may assume that $r' = r \in J$ and $s' = s \in \widetilde{J}'$. Also, we may assume that $\pi(r)\pi(s)$ has finite order $n > 1$ in W, without loss of generality. We have by (2) that either $\pi(s) = u\pi(t)u^{-1}$ for $u \in W'_{I\cap r^\perp}$, $t \in J$ with $t \neq r$ and $\pi(t)\pi(r)$ of finite order, or $\pi(s) = uv\pi(r)vu^{-1}$ for some $u \in W'_{I\cap r^\perp}$ and $v \in I$ with $v\pi(r)$ of finite order greater than 2. In the first (resp., second) case, $\pi(r)\pi(s) = u\pi(r)\pi(t)u^{-1}$ (resp., $\pi(r)\pi(s) = u\pi(r)v\pi(r)vu^{-1}$)

and n is the order of $\pi(r)\pi(t)$ (resp., half the order of $\pi(r)v$) in W. In the first case, $s = utu^{-1}$. The relations of (U, S) imply that $rs = urtu^{-1}$, which has the same order as rt in U. In the second case, $s = uvrvu^{-1}$ and the relations of (U, S) imply that $rs = urvrvu^{-1}$, which has order equal to half the order of rv in U. The definition of U implies that the order of rt (resp., rv) in U is the same as that of $\pi(r)\pi(t)$ (resp., $\pi(r)v$) in W and so the order of rs in U is equal to the order n of $\pi(r)\pi(s)$ in W in either case, completing the proof. □

Remark - We leave open the question of whether different choices of the set J of W_I-orbit representatives satisfying the conditions in Theorem 2.1 are possible, or if possible, would give rise to isomorphic Coxeter systems $(W, I \cup J)$.

3. Semi-direct products and root systems

In this section, we use the standard geometric realization of (W, S) as a reflection group associated to a based root system. In fact, it is convenient (and essential for the main result Theorem 3.11 of this section) to introduce a slightly more general class of geometric realizations with better "functoriality" properties with respect to inclusions of reflection subgroups.

Let $\mathscr{E}$ be a $\mathbb{R}$-vector space equipped with a symmetric $\mathbb{R}$-bilinear form $\langle,\rangle$. We say a subset Π of $\mathscr{E}$ is positively independent if $\sum_{\alpha\in\Pi} c_\alpha \alpha = 0$ with all $c_\alpha \geq 0$ implies that all $c_\alpha = 0$. For example, any $\mathbb{R}$-linearly independent set is positively independent. If $\alpha \in \mathscr{E}$ is such that $\langle \alpha, \alpha \rangle = 1$, we set $\alpha^\vee = 2\alpha$ and we define

$$\begin{array}{rccc} s_\alpha : & \mathscr{E} & \longrightarrow & \mathscr{E} \\ & v & \longmapsto & v - \langle v, \alpha^\vee \rangle \alpha. \end{array}$$

Then s_α is an orthogonal reflection (with respect to $\langle,\rangle$). Let

$$\mathrm{COS} = \{\cos(\pi/m) \mid m \in \mathbb{N}_{\geqslant 2}\} \cup \mathbb{R}_{\geqslant 1}. \tag{3.6}$$

Assume that Π is a subset of $\mathscr{E}$ with the following properties (i)–(iii):

(i) Π is positively independent.
(ii) For all $\alpha \in \Pi$, $\langle \alpha, \alpha \rangle = 1$.
(iii) For all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$, one has $\langle \alpha, \beta \rangle \in -\mathrm{COS}$.

Let $S := \{ s_\alpha \mid \alpha \in \Pi \}$, let W be the subgroup of the orthogonal group $O(\mathscr{E}, \langle,\rangle)$ generated by S,

$$\Phi := \{w(\alpha) \mid w \in W \text{ and } \alpha \in \Pi\}, \qquad \Phi_+ = \Phi \cap \Bigl(\sum_{\alpha\in\Pi} \mathbb{R}_{\geqslant 0}\, \alpha\Bigr).$$

Then (W, S) is a Coxeter system, in which the order m_{s_α,s_β} of the product $s_\alpha s_\beta$ for $\alpha, \beta \in \Pi$ is given by

$$m_{s_\alpha,s_\beta} = \begin{cases} m, & \text{if } \langle\, \alpha, \beta\, \rangle = -\cos\dfrac{\pi}{m}, \quad m \in \mathbb{N}_{\geqslant 1} \\ \infty, & \text{if } \langle\, \alpha, \beta\, \rangle \leqslant -1. \end{cases} \tag{3.7}$$

One has

$$\Phi = \Phi_+ \,\dot\cup\, -\Phi_+. \tag{3.8}$$

When the above conditions hold, we say that (Φ, Π) is a based root system in $(\mathscr{E}, \langle , \rangle)$ with associated Coxeter system (W, S). Every Coxeter system is isomorphic to the Coxeter system of some based root system (and even to one with $\langle\, \alpha, \beta\, \rangle = -\cos\frac{\pi}{m_{s_\alpha,s_\beta}}$ for all $\alpha, \beta \in \Pi$, and with Π a basis of $\mathscr{E}$; a based root system of this type is called a standard based root system). All the usual results for standard based root systems which we use in this paper, and their proofs, extend mutatis mutandis to the based root systems as defined above, unless additional hypotheses are indicated in our statements here (as in Lemma 3.3 below, for example).

Let us collect some additional basic facts about such based root systems.

Lemma 3.1. *For $w \in W$ and $\alpha \in \Phi_+$, one has $w(\alpha) \in \Phi_+$ iff $\ell(ws_\alpha) > \ell(w)$.*

Lemma 3.2. *Let $\Delta \subseteq \Phi_+$, let $T' = \{s_\alpha \mid \alpha \in \Delta\}$ and let W' denote the subgroup of W generated by T'. Then T' is the set of canonical Coxeter generators of W' if and only if $-\langle \alpha, \beta \rangle \in \mathrm{COS}$ for all α, $\beta \in \Delta$ such that $\alpha \neq \beta$.*

Proof. See [6, (4.4)] □

Lemma 3.3 (Brink). *Let $\gamma \in \Phi_+$. Then one may write $\gamma = \sum_{\alpha\in\Pi} c_\alpha \alpha$ with $c_\alpha/2 \in \mathrm{COS}$ for all $\alpha \in \Pi$. In particular, if $c_\alpha \notin \{0, 1\}$, then $c_\alpha \geqslant \sqrt{2}$. If Π is linearly independent, the c_α are uniquely determined by the conditions $\gamma = \sum_{\alpha\in\Pi} c_\alpha \alpha$ and $c_\alpha \in \mathbb{R}$.*

Proof. For the standard reflection representation, for which Π is linearly independent, see [5, Proposition 2.1]. A quick sketch in general is as follows. One checks the statement for dihedral Coxeter systems (for which Π is automatically linearly independent) by direct calculations (see [6, (4.1)]). Then in general, a standard proof (loc cit) of Lemma 3.1 by reduction to rank two shows that there is some choice of root coefficients c_α such that all c_α are expressible as polynomials with non-negative integer coefficients in the (positive) root coefficients for rank two standard parabolic subgroups, and the result follows. □

Lemma 3.4. *Let $\beta \in \Pi$ and $\alpha \in \Phi_+ \setminus \{\beta\}$. Then*

(a) *$s_\beta(\alpha) \in \Phi_+$ and $s_{s_\beta(\alpha)} = s_\beta s_\alpha s_\beta$.*

(b) $\ell(s_\beta s_\alpha s_\beta)$ *is equal to* $\ell(s_\alpha)+2$, $\ell(s_\alpha)$ *or* $\ell(s_\alpha)-2$ *according as whether* $\langle \alpha, \beta \rangle < 0$, $\langle \alpha, \beta \rangle = 0$ *or* $\langle \alpha, \beta \rangle > 0$. *If* $\langle \alpha, \beta \rangle = 0$, *then* $s_\beta s_\alpha s_\beta = s_\alpha$.

Proof. Part (a) is well-known, and so is (b) in the special case of linearly independent simple roots. One may also verify (b) for dihedral Coxeter systems by direct calculation (using [6, (4.1)]) again, for instance). In general, (b) may be reduced to the dihedral case as follows. Let $W' := \langle s_\alpha, s_\beta \rangle$, $T' = \chi(W')$ and l' be the length function of (W', T'). In case $\langle \alpha, \beta \rangle = 0$, then by the dihedral case, $s_\beta s_\alpha = s_\alpha s_\beta$ and so $\ell(s_\beta s_\alpha s_\beta) = \ell(s_\alpha)$. In case $\langle \alpha, \beta \rangle < 0$, then by the dihedral case, one has $\ell'(s_\beta) < \ell'(s_\beta s_\alpha) < \ell'(s_\beta s_\alpha s_\beta)$. Hence by Lemma 1.2 (c), one has $\ell(s_\beta) < \ell(s_\beta s_\alpha) < \ell(s_\beta s_\alpha s_\beta)$ and thus $\ell(s_\beta s_\alpha s_\beta) = \ell(s_\alpha) + 2$ as required. The remaining case $\langle \alpha, \beta \rangle > 0$ follows from (a) and the second case applied to $\alpha' := s_\beta(\alpha)$ in place of α, since $\langle \alpha', \beta \rangle < 0$. $\square$

The chief technical advantage of the class of based root systems is explained by Lemma 3.5 below. It follows from the definition and previously given facts about based root systems (especially Lemma 3.2 and (3.7)).

Lemma 3.5. *Let* (Φ, Π) *be a based root system in* $(\mathscr{E}, \langle , \rangle)$, *with associated Coxeter system* (W, S). *Let* W' *be a reflection subgroup of* (W, S) *and set* $S' := \chi(W')$. *Let* $\Psi := \{ \alpha \in \Phi \mid s_\alpha \in W' \}$ *and* $\Delta := \{ \alpha \in \Phi_+ \mid s_\alpha \in S' \}$. *Then* (Ψ, Δ) *is a based root system in* $(\mathscr{E}, \langle , \rangle)$ *with associated Coxeter system* (W', S').

REMARK - Note that even if (Φ, Π) is a standard based root system and S' is finite, the elements of Δ need not be linearly independent, and for elements α, β of Δ such that $s_\alpha s_\beta$ has infinite order, one may have $\langle \alpha, \beta \rangle < -1$. Thus, the lemma fails for the class of standard based root systems in two important respects.

Although not logically required in this paper, we include the following alternative proof of Theorem 1.8 and part of Theorem 1.1 using based root systems, because of its intrinsic interest and since the general method of proof may be applicable in other situations. Precisely, we shall prove here the following:

Theorem 3.6. *Let* (W, S) *be a Coxeter system. Let* $S = I \dot\cup J$ *be a partition of* S *as in* Theorem 1.1, *and define* $\widetilde{W}, \widetilde{J}, \widetilde{M}$ *as in* Theorems 1.1 *and* 1.8. *Then* $(\widetilde{W}, \widetilde{J})$ *is a Coxeter system with Coxeter matrix* $\widetilde{M}$ *and* $\widetilde{J} = \chi(\widetilde{W})$ *is the canonical set of Coxeter generators of* $\widetilde{W}$.

Proof. We assume without loss of generality that (W, S) is the Coxeter system associated to a based root system (Φ, Π) such that Π is linearly independent. We keep other notation as above.

Let $\Pi_K := \{ \alpha \in \Pi \mid s_\alpha \in K \}$ for any $K \subseteq S$. By (1.1) and (3.7), the assumption that no element of I is conjugate to any element of J is therefore equivalent to the assertion that if $\gamma \in \Pi_I$ and $\delta \in \Pi_J$, then $\langle \gamma, \delta \rangle$ is either of the form $\langle \gamma, \delta \rangle = -\cos \frac{\pi}{2m}$ for some $m \in \mathbb{N}_{\geq 1}$ or satisfies $\langle \gamma, \delta \rangle \leqslant -1$. In particular,

$$\text{(3.9)} \qquad \textit{If } \gamma \in \Pi_I \textit{ and } \delta \in \Pi_J, \textit{ then } \langle \gamma, \delta \rangle \leqslant -\frac{\sqrt{2}}{2}.$$

Now, let

$$\widetilde{\Pi} = \{w(\alpha) \mid w \in W_I \text{ and } \alpha \in \Pi_J\}.$$

Then $\widetilde{\Pi} \subseteq \Phi_+$ by Lemma 3.1, and $\widetilde{J} = \{s_\alpha \mid \alpha \in \widetilde{\Pi}\}$.

By Lemma 3.2 and (3.7), it is sufficient to show that, if $\tilde{\alpha}$, $\tilde{\beta} \in \widetilde{\Pi}$ are such that $\tilde{\alpha} \neq \tilde{\beta}$ and if $\tilde{s} = s_{\tilde{\alpha}}$ and $\tilde{t} = s_{\tilde{\beta}}$, then

$$(*) \qquad \begin{cases} \langle \tilde{\alpha}, \tilde{\beta} \rangle = -\cos\left(\dfrac{\pi}{\tilde{m}_{\tilde{s},\tilde{t}}}\right) & \text{if } \tilde{m}_{\tilde{s},\tilde{t}} < \infty, \\ \langle \tilde{\alpha}, \tilde{\beta} \rangle \leqslant -1 & \text{if } \tilde{m}_{\tilde{s},\tilde{t}} = \infty, \end{cases}$$

For this, let $s = \nu(\tilde{s})$, $t = \nu(\tilde{t})$ and let x, $y \in W_I$ be such that $\tilde{s} = xsx^{-1}$ and $\tilde{t} = yty^{-1}$. Let $\alpha = x^{-1}(\tilde{\alpha})$, $\beta = y^{-1}(\tilde{\beta})$ and $w = f(\tilde{s}, \tilde{t})$. Then α, $\beta \in \Pi_J$, $s = s_\alpha$, $t = s_\beta$ and

$$\langle \tilde{\alpha}, \tilde{\beta} \rangle = \langle \alpha, w(\beta) \rangle.$$

Indeed, if we write $x^{-1}y = awb$ with $a \in W_{I \cap s^\perp}$ and $b \in W_{I \cap t^\perp}$, then

$$\langle \tilde{\alpha}, \tilde{\beta} \rangle = \langle x(\alpha), y(\beta) \rangle = \langle \alpha, awb(\beta) \rangle = \langle a^{-1}(\alpha), wb(\beta) \rangle = \langle \alpha, w(\beta) \rangle.$$

We shall now need the notion of the support of a positive root. If $\delta \in \Phi_+$, write $\delta = \sum_{\gamma \in \Pi} c_\gamma \gamma$ with $c_\gamma \geqslant 0$: the *support* $\operatorname{supp}(\delta)$ of δ is the subset of Π defined by $\operatorname{supp}(\delta) := \{ \gamma \in \Pi \mid c_\gamma \neq 0 \}$. This is well-defined since we have assumed Π is linearly independent. We recall the following facts:

Lemma 3.7. *Let $\delta \in \Phi_+$ and $A := \{ s_\gamma \mid \gamma \in \operatorname{supp}(\delta) \}$. Then*
$s_\delta \in W_A$.
(2) *The full subgraph of the Coxeter graph of (W, S) with vertex set A is connected.*

Proof. We prove (a)–(b) by induction on $l(s_\delta)$. If $l(s_\delta) = 1$, then $\delta \in \Pi$ and (a)–(b) are clear. Otherwise, write $\delta = \sum_{\alpha \in \Pi} c_\alpha \alpha$ with all $c_\alpha \geq 0$. Since $0 < 1 = \langle \alpha, \delta \rangle = \sum_\alpha c_\alpha \langle \alpha, \delta \rangle$ there is some $\alpha \in \operatorname{supp}(\delta)$ with $\langle \alpha, \delta \rangle > 0$. Note $\alpha \neq \delta$ since $\delta \notin \Pi$, so $\gamma := s_\alpha(\delta) \in \Phi_+$. By Lemma 3.4, $l(s_\gamma) = l(s_\delta) - 2$. Let $B := \{ s_\beta \mid \beta \in \operatorname{supp}(\delta) \}$. By induction, $s_\gamma \in W_B$ and the full subgraph of the Coxeter graph of (W, S) on vertex set B is connected. Since $\delta = s_\alpha(\gamma) = \gamma + \langle \delta, \alpha \rangle \alpha$, we have $\operatorname{supp}(\delta) = \operatorname{sup}(\gamma) \cup \{\alpha\}$ and $A = B \cup \{s_\alpha\}$. Since $0 > -\langle \alpha, \delta \rangle = \langle \alpha, \gamma \rangle$, an argument like that above shows that there is some $\beta \in \operatorname{supp}(\gamma)$ with $\langle \alpha, \beta \rangle < 0$. Therefore s_α is joined by an edge in the Coxeter graph of (W, S) to $s_\beta \in B$, completing the inductive proof of (b). Since $s_\delta = s_\alpha s_\gamma s_\alpha \in W_A$, the inductive proof of (a) is also finished $\square$

Now, let Γ be the unique subset of Π_I such that $\operatorname{supp}(w(\beta)) = \Gamma \cup \{\beta\}$ and set $I_\Gamma = \{s_\gamma \mid \gamma \in \Gamma\}$. We write

$$w(\beta) = \beta + \sum_{\gamma\in\Gamma} c_\gamma \gamma,$$

with $c_\gamma > 0$. In order to prove $(*)$, we shall need the following lemmas:

Lemma 3.8. *Let $\gamma \in \Pi_I$. Then:*
If $\gamma \in \Gamma$, then $c_\gamma \geq \sqrt{2}$.
(b) *If s_γ appears in a reduced expression for w and $\langle \beta, \gamma^\vee \rangle \neq 0$, then $\gamma \in \Gamma$ and either $c_\gamma = -\langle \beta, \gamma^\vee \rangle$ or $c_\gamma \geq 2\sqrt{2}$.*

Proof. We shall argue by induction on $\ell(w)$. If $\ell(w) = 0$, this is vacuously true. Otherwise, write $w = xs_\delta$ where $\delta \in \Pi_I$ and $\ell(x) < \ell(w)$. We have $s_\delta(\beta) = \beta + c\delta$ where $c := -\langle \beta, \delta^\vee \rangle$. If $c = 0$, then $w(\beta) = x(\beta)$ and the desired result follows by induction. Otherwise, $c \geq \sqrt{2}$ and $w(\beta) = x(\beta) + cx(\delta)$. Note $x(\delta) \in \Phi_+$ by Lemma 3.1 since $\ell(xs_\delta) > \ell(x)$. Using the inductive hypothesis (a)–(b) for $x(\beta)$ and Lemma 3.3 for $x(\delta)$, one gets (a)–(b) for $w(\beta)$ (for (b), one has to consider the cases $\gamma = \delta$, $\gamma \neq \delta$ separately, and note that if s_δ does not appear in a reduced expression for x, then the coefficient of δ in $x(\delta)$ is 1). $\square$

Lemma 3.9. *If $I_\Gamma \subseteq s^\perp$, then $w = 1$.*

Proof. Indeed, if $I_\Gamma \subseteq s^\perp$, then Lemma 3.7(a) implies that we have $wtw^{-1} \in W_{\{t\}\cup(I\cap s^\perp)}$. In other words, $wt \in W_{\{t\}\cup(I\cap s^\perp)}w$. But w has minimal length in $W_{\{t\}\cup(I\cap s^\perp)}w$ by construction, so wt does not have minimal length in $W_{\{t\}\cup(I\cap s^\perp)}wt$. By Deodhar's Lemma, there exists $u \in \{t\} \cup (I \cap s^\perp)$, such that $wt = uw$. In other words, $u = wtw^{-1}$ and, since no element of I is conjugate to t, we have $u = t$ and $wt = tw$. So $w \in W_{I\cap t^\perp}$ (see Lemma 1.5), and thus $w = 1$ because w has minimal length in $wW_{I\cap t^\perp}$. $\square$

We shall now prove $(*)$ by a case-by-case analysis:

• If $s = t$ and $w \in I$, let us write $w = s_\gamma$ with $\gamma \in \Pi_I$. Then $\alpha = \beta$, $\tilde{m}_{\tilde{s},\tilde{t}} = m_{s,w}/2$ and $w(\beta) = \alpha - \langle \alpha, \gamma^\vee \rangle \gamma$, so

$$\langle \alpha, w(\beta) \rangle = \langle \alpha, \alpha \rangle - 2\langle \alpha, \gamma \rangle^2 = 1 - 2\cos^2\Big(\frac{\pi}{m_{s,w}}\Big) = -\cos\Big(\frac{2\pi}{m_{s,w}}\Big),$$

as required.

• If $s = t$ and $\ell(w) \geqslant 2$, then $\tilde{m}_{\tilde{s},\tilde{t}} = \infty$. First, note that

$$I_\Gamma \not\subseteq s^\perp$$

(see Lemma 3.9). Moreover,

$$\langle \alpha, w(\beta) \rangle = \langle \alpha, \beta \rangle + \sum_{\gamma\in\Gamma} c_\gamma \langle \alpha, \gamma \rangle = 1 + \sum_{\substack{\gamma\in\Gamma \\ s_\gamma \notin s^\perp}} c_\gamma \langle \alpha, \gamma \rangle$$

But, if $\gamma \in \Gamma$ is such that $s_\gamma \notin s^\perp$, then $c_\gamma \geqslant \sqrt{2}$ by Lemma 3.8 (a) and $\langle\, \alpha, \gamma \,\rangle = -\cos(\pi/m_{s,s_\gamma}) \leqslant -\sqrt{2}/2$ by (3.9) (since $\alpha \in \Pi_J$ and $\gamma \in \Pi_I$). Therefore,

$$\langle\, \alpha, w(\beta) \,\rangle \leqslant 1 - |I_\Gamma \setminus s^\perp|.$$

So, if $|I_\Gamma \setminus s^\perp| \geqslant 2$, then $\langle\, \alpha, w(\beta) \,\rangle \leqslant -1$, as required.

So we may assume that $I_\Gamma \setminus s^\perp = \{s_\gamma\}$ with $\gamma \in \Gamma$. Note that $\langle\, \alpha, w(\beta) \,\rangle = 1 - c_\gamma \langle\, \alpha, \gamma \,\rangle$ and that s_γ appears in a reduced expression of w. By Lemma 3.8 (b), two cases may occur:

- If $c_\gamma \geqslant 2\sqrt{2}$ then, since $\langle\, \alpha, \gamma \,\rangle \leqslant -\sqrt{2}/2$ (again by the inequality (3.9)), we get that $\langle\, \alpha, w(\beta) \,\rangle \leqslant -1$, as required.
- If $c_\gamma = -\langle\, \beta, \gamma^\vee \,\rangle$ then

$$\operatorname{supp}(s_\gamma w\beta) = \operatorname{supp}(w\beta) \setminus \{\gamma\} = (\Gamma \setminus \{\gamma\}) \cup \{\beta\}.$$

But no element of $\{\, s_\delta \mid \delta \in \Gamma \setminus \{\gamma\} \,\}$ is connected to s_β in the Coxeter graph of (W, S), so by Lemma 3.7 (b) we get that $\Gamma = \{\gamma\}$, $\operatorname{supp}(s_\gamma w\beta) = \{\beta\}$ and so $s_\gamma w\beta = \beta$. Hence $s_\gamma w \in W_{I \cap t^\perp}$. By Deodhar's Lemma, this can only happen if $w = s_\gamma$, which contradicts the fact that $\ell(w) \geqslant 2$.

• If $s \neq t$ and $w = 1$, then $\tilde{m}_{\tilde{s},\tilde{t}} = m_{s,t}$ and

$$\langle\, \tilde{\alpha}, \tilde{\beta} \,\rangle = \langle\, \alpha, \beta \,\rangle = -\cos\Bigl(\frac{\pi}{m_{s,t}}\Bigr),$$

as required.

• If $s \neq t$ and $w \neq 1$, then $\tilde{m}_{\tilde{s},\tilde{t}} = \infty$. First, note that

$$I_\Gamma \nsubseteq s^\perp$$

(see Lemma 3.9). So let $\gamma \in \Gamma$ be such that $\langle\, \alpha, \gamma \,\rangle \neq 0$. Then $c_\gamma \geqslant \sqrt{2}$ by Lemma 3.8 and, by (3.9), we have $\langle\, \alpha, \gamma \,\rangle \leqslant -\sqrt{2}/2$ (since $\alpha \in \Pi_J$ and $\gamma \in \Pi_I$). So

$$\langle\, \alpha, w(\beta) \,\rangle \leqslant \langle\, \alpha, \beta \,\rangle - 1 + \sum_{\gamma' \neq \gamma} c_{\gamma'} \langle\, \alpha, \gamma' \,\rangle \leqslant -1$$

because $\langle\, \alpha, \beta \,\rangle \leqslant 0$ and $\langle\, \alpha, \gamma' \,\rangle \leqslant 0$ for all $\gamma' \in \Pi_I$.

The proof of Theorem 3.6 is now complete. □

The final main result of this section is a geometric variant (Theorem 3.11 below) of Theorem 2.1. To formulate it, we shall require the notions of automorphisms, fundamental chamber and Tits cone of a based root system. The latter two are principally of interest when the form $\langle , \rangle$ on $\mathscr{E}$ is non-degenerate, but our application won't require this (and non-degeneracy can always be achieved by enlarging the space $\mathscr{E}$ and extending the form $\langle , \rangle$, anyway).

Let (Φ, Π) be a based root system in $(\mathscr{E}, \langle , \rangle)$, with associated Coxeter system (W, S). By an automorphism of (Φ, Π), we mean an element θ of $O(\mathscr{E}, \langle , \rangle)$ which restricts to permutations of both Π and Φ. For example, in the setting of the proof of Theorem 3.6, W_I acts naturally as a group of based root system automorphisms of the based root system attached by Lemma 3.5 to $\widetilde{W}$.

In general, we define the fundamental chamber of (W, S) on $\mathscr{E}$ to be the subset $\mathscr{C} = \mathscr{C}_{(W,S)} := \{ \rho \in \mathscr{E} \mid \langle \alpha, \rho \rangle \geq 0 \text{ for all } \alpha \in \Pi \}$ of $\mathscr{E}$, and we call $\mathscr{X} = \mathscr{X}_{(W,S)} = W\mathscr{C} := \cup_{w \in W} w(\mathscr{C})$ the Tits cone. The most basic properties of $\mathscr{C}$ and $\mathscr{X}$ (see [4]) are recalled in the following Lemma.

Lemma 3.10. (a) $\mathscr{X} = \{ \rho \in \mathscr{E} \mid |\{ \alpha \in \Phi_+ \mid \langle \alpha, \rho \rangle < 0 \}| < \infty \}$. *In particular, $\mathscr{X}$ is a convex cone in $\mathscr{E}$.*
(b) *Any W-orbit on X contains a unique element of $\mathscr{C}$.*
(c) *For $\alpha \in \mathscr{C}$, the stabilizer $W_\alpha := \{ w \in W \mid w(\alpha) = \alpha \}$ of α is the standard parabolic subgroup of W generated by $\{ s \in S \mid s(\alpha) = \alpha \}$.*

Now we may state:

Theorem 3.11. *Let (Ψ, Δ) and $(\widetilde{\Phi}, \widetilde{\Pi})$ be two based root systems in $(\mathscr{E}, \langle , \rangle)$ with associated Coxeter systems (W', I) and $(\widetilde{W}, \widetilde{J})$ respectively. Let $\mathscr{C} := \mathscr{C}_{(W',I)}$ and $\mathscr{X} := \mathscr{X}_{(W',I)}$. Assume that $W'(\widetilde{\Pi}) \subseteq \widetilde{\Pi}$. Then W' acts as a group of based root system automorphisms of $(\widetilde{\Phi}, \widetilde{\Pi})$ and also as a group of automorphisms of the Coxeter system $(\widetilde{W}, \widetilde{J})$. Let W denote the subgroup of $O(\mathscr{E}, \langle , \rangle)$ generated by the subset $W' \cup \widetilde{W}$. Then $W = \widetilde{W} \rtimes W'$. Under these assumptions, the following conditions are equivalent:*

(i) *There is a based root system (Φ, Π) with $\Delta \subseteq \Pi \subseteq \Delta \cup \widetilde{\Pi}$ and $\widetilde{\Pi} = W'(\Pi \setminus \Delta)$.*
(ii) *$\Delta \cup \widetilde{\Pi}$ is positively independent and $\widetilde{\Pi} \subseteq -\mathscr{X}$.*

Assume conditions (i)–(ii) *hold. Then $\Pi = \Delta \,\dot{\cup}\, (\widetilde{\Pi} \cap -\mathscr{C})$ (so (Φ, Π) is uniquely determined in* (i)*), $\Psi \cup \widetilde{\Phi} \subseteq \Phi$, and $\widetilde{\Phi}_+ \subseteq -\mathscr{X}$. Set $S := \{ s_\alpha \mid \alpha \in \Pi \}$ and $J = S \setminus I$. Then (W, S) is the Coxeter system associated to the based root system (Φ, Π), $\widetilde{J} = \{ wsw^{-1} \mid w \in W', s \in J \}$, and no element of I is conjugate to any element of J. The semidirect product decomposition $W = \widetilde{W} \rtimes W'$ is that attached by* Theorem 1.1 *to the subsets I and J of S.*

Proof. For any $\theta \in O(\mathscr{E}, \langle , \rangle)$ and $\alpha \in \mathscr{E}$ with $\langle \alpha, \alpha \rangle = 1$, one has $\langle \theta(\alpha), \theta(\alpha) \rangle = 1$ and $s_{\theta(\alpha)} = \theta s_\alpha \theta^{-1}$. Assume further that $\theta(\Pi') \subseteq \Pi'$. Then this implies that $\widetilde{J}$, and hence $\widetilde{W}$, is stable under conjugation by θ, and so θ acts as an automorphism of $(\widetilde{W}, \widetilde{J})$. If $\alpha \in \widetilde{\Phi}$, we can write $\alpha = x(\beta)$ for some $\beta \in \widetilde{\Pi}$ and $x \in \widetilde{W}$. Then $\theta(\alpha) = \theta x(\beta) = (\theta x \theta^{-1})(\theta(\beta)) \in \widetilde{\Phi}$ since $\theta x \theta^{-1} \in \widetilde{W}$ and $\theta(\beta) \in \widetilde{\Pi}$. Hence $\theta(\widetilde{\Phi}) \subseteq \widetilde{\Phi}$. For

$\gamma \in \widetilde{\Phi}_+$, we may write $\gamma = \sum_{\alpha\in\widetilde{\Pi}} c_\alpha\alpha$ with all $c_\alpha \geq 0$. Then $\theta(\gamma) = \sum_{\alpha\in\widetilde{\Pi}} c_\alpha\theta(\alpha) \in \widetilde{\Phi}_+$ since all $\theta(\alpha) \in \widetilde{\Pi}$, showing that $\theta(\widetilde{\Phi}_+) \subseteq \widetilde{\Phi}_+$.

The above all applies with $\theta \in W'$, proving that W' acts as automorphisms of $(\widetilde{W}, \widetilde{J})$ and $(\widetilde{\Phi}, \widetilde{\Pi})$. In particular, W' normalizes $\widetilde{W}$. If $w \in W'$, then w permutes $\widetilde{\Phi}_+$. If $w \in W' \cap \widetilde{W}$, this implies that $\tilde{\ell}(w) = 0$ (since w makes no element of $\widetilde{\Phi}_+$ negative) so $w = 1_{W'}$. From the above, we see that $W = W'\widetilde{W} = \widetilde{W} \rtimes W'$ as claimed. We also see that $\Psi \cap \widetilde{\Phi} = \emptyset$, for if $\alpha \in \Psi \cap \widetilde{\Phi}$, then $s_\alpha \in W' \cap \widetilde{W} = \{1_{W'}\}$ which is a contradiction. From this, one sees further that $\widetilde{\Phi}$ is stable under the W-action on $\mathscr{E}$ and hence that no element of Ψ is W-conjugate to any element of $\widetilde{\Phi}$.

Now suppose that the assumptions of (i) hold. Since Π_+ is positively independent, it follows that Φ_+ is positively independent, and hence so also is the subset $\Delta \cup \widetilde{\Pi}$ of Φ_+. Let $\alpha \in \Pi \setminus \Delta \subseteq \widetilde{\Pi}$. Since $\alpha \notin \Delta$, we have $\langle \alpha, \beta \rangle \in -\mathrm{COS}$ for all $\beta \in \Delta$. In particular, $\langle \alpha, \beta \rangle \leqslant 0$ so $\alpha \in -\mathscr{C}$. Thus, $\Pi \setminus \Delta \subseteq -\mathscr{C}$. Hence

$$\widetilde{\Pi} = W'(\Pi \setminus \Delta) \subseteq W'(-\mathscr{C}) = -\mathscr{X}.$$

Therefore $\widetilde{\Phi}_+ \subseteq -\mathscr{X}$ also since $\mathscr{X}$ is a convex cone. Since every W'-orbit on $-\mathscr{X}$ contains a unique point of $-\mathscr{C}$, $\widetilde{\Pi}$ is W'-stable and $\widetilde{\Pi} \subseteq W'(\Pi \setminus \Delta)$, it follows using Lemma 3.10 (b) that $\Pi \setminus \Delta = \widetilde{\Pi} \cap -\mathscr{C}$. Observe also that we have $\Psi \cup \widetilde{\Phi} \subseteq \Phi$ and so

$$W = \langle s_\alpha \mid \alpha \in \Psi \cup \widetilde{\Phi} \rangle \subseteq \langle s_\alpha \mid \alpha \in \Phi \rangle = \langle s_\alpha \mid \alpha \in \Pi \rangle \subseteq \langle s_\alpha \mid \alpha \in \Delta \cup \widetilde{\Pi} \rangle = W$$

which implies that if (i) holds, then the Coxeter system associated to (Φ, Π) is (W, S) where $S := \{ s_\alpha \mid \alpha \in \Pi \}$.

Now suppose that the assumptions of (ii) hold. Set $\Pi = \Delta \mathbin{\dot\cup} (\widetilde{\Pi} \cap -\mathscr{C})$. Clearly, $\Delta \subseteq \Pi \subseteq \Delta \cup \widetilde{\Pi}$. We also have $\widetilde{\Pi} = W'(\Pi \setminus \Delta)$ since $\widetilde{\Pi} \subseteq -\mathscr{X}$ and $\widetilde{\Pi}$ is W'-stable. Let $S := \{ s_\alpha \mid \alpha \in \Pi \}$ and W'' be the subgroup generated by S. It is clear W'' contains W' and s_α for $\alpha \in \Pi \setminus \Delta$, so it also contains $ws_\alpha w^{-1}$ for such α and all $w \in W'$. That is, W'' contains the group generated by s_β for all $\beta \in W'(\Pi \setminus \Delta) = \widetilde{\Pi}$. So $W'' \supseteq W'\widetilde{W} = W$. But clearly, $S \subseteq W$, so $W'' = W$. Let $\Phi = W\Pi$.

Since $\Delta \cup \widetilde{\Pi}$ is positively independent, to show that (Φ, Π) is a based root system, it will suffice to show that if $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$, then $c := -\langle \alpha, \beta \rangle \in \mathrm{COS}$. If both α, β are in Δ, or both are in $\widetilde{\Pi}$, this follows since (Ψ, Δ) and $(\widetilde{\Phi}, \widetilde{\Pi})$ are based root systems. The remaining case is that, say, $\alpha \in \Delta$ and $\beta \in \widetilde{\Pi}$. We show that in this case, $c \in \mathrm{COS}' := \{-\cos \pi/2m \mid m \in \mathbb{N}_{\geq 1}\} \cup \mathbb{R}_{\geq 1}$. We have $c \geq 0$ since $\beta \in -\mathscr{C}$. Also, $s_\alpha(\beta) = \beta + 2c\alpha \in \widetilde{\Pi}$. If $s_\alpha(\beta) = \beta$, then $c = 0 \in \mathrm{COS}'$. Otherwise, $s_\alpha(\beta) \neq \beta$ are both in $\widetilde{\Pi}$, so $d := -\langle s_\alpha(\beta), \beta \rangle \in \mathrm{COS}$ because $(\widetilde{\Phi}, \widetilde{\Pi})$ is a based root system. But $d = -\langle \beta + 2c\alpha, \beta \rangle = -1 + 2c^2$. So $c = \sqrt{\frac{d+1}{2}}$ with $d \in \mathrm{COS}$. If

$d \geq 1$, say $d = \cosh \lambda$ where $\lambda \in \mathbb{R}$, then $c = \cosh \frac{\lambda}{2} \geq 1$ so $c \in \mathrm{COS}'$. Otherwise, $d = \cos \frac{\pi}{m}$ for some $m \in \mathbb{N}_{\geq 2}$, so $c = \cos \frac{\pi}{2m} \in \mathrm{COS}'$. This shows that (ii) implies (i). Note that $J = S \setminus I = \{\, s_\alpha \mid \alpha \in \Pi \setminus \Delta \,\}$. The argument above also shows that no element of I is W-conjugate to any element of J.

Assuming that (i) and (ii) both hold, the remaining assertions of the Theorem follow directly from the consequences of (i)–(ii) proved above. □

4. Affine reflection groups

Let E be a finite dimensional affine space over $\mathbb{R}$ and assume that the underlying vector space $\mathscr{E}$ is endowed with a positive definite scalar product $\langle\, , \,\rangle$. If H is an hyperplane in E, we denote by s_H the orthogonal reflection with respect to H.

Let $\mathfrak{A}$ be an (affine) hyperplane arrangement in E and let W be the subgroup of $O(E, \langle\, , \,\rangle)$ generated by $(s_H)_{H \in \mathfrak{A}}$. As in [4, Chapter V, §3], we assume that the following hypothesis are satisfied:

(D1) W stabilizes $\mathfrak{A}$.
(D2) The group W, endowed with the discrete topology, acts properly on E.

We can then define the notions of $\mathfrak{A}$-chambers, $\mathfrak{A}$-walls, $\mathfrak{A}$-facets, $\mathfrak{A}$-faces as defined in [4, Chapter V, §1]. We fix an $\mathfrak{A}$-chamber C and we denote by Δ the set of $\mathfrak{A}$-walls of C. Let $S = \{s_H \mid H \in \Delta\}$. Then (W, S) is a Coxeter system and $\overline{C}$ (the closure of C) is a fundamental domain for the action of W on E (see [4, Chapter V, §3, Theorems 1 and 2]).

We still assume that we have a partition $S = I \,\dot\cup\, J$ such that no element in I is W-conjugate to an element in J and we keep the notation of the previous sections. We set

$$T = \{s_H \mid H \in \mathfrak{A}\}, \quad \widetilde{\Delta} = \{H \in \mathfrak{A} \mid s_H \in \widetilde{J}\},$$
$$\widetilde{T} = T \cap \widetilde{W} \quad \text{and} \quad \widetilde{\mathfrak{A}} = \{H \in \mathfrak{A} \mid s_H \in \widetilde{T}\}.$$

Then $\widetilde{\mathfrak{A}}$ is an hyperplane arrangement satisfying (D1) and (D2). Let $\widetilde{C}$ be the unique $\widetilde{\mathfrak{A}}$-chamber containing C. Then $\widetilde{\Delta}$ is the set of $\widetilde{\mathfrak{A}}$-walls of $\widetilde{C}$. We have:

Proposition 4.1. $\overline{\widetilde{C}} = \displaystyle\bigcup_{w \in W_I} w(\overline{C})$.

Proof. Let $\hat{C} = \displaystyle\bigcup_{w \in W_I} w(\overline{C})$. First, note that W_I stabilizes $\widetilde{\mathfrak{A}}$, so W_I stabilizes $\widetilde{C}$ (and $\overline{\widetilde{C}}$). Therefore, $\hat{C} \subseteq \overline{\widetilde{C}}$.

Conversely, let $p \in \overline{\widetilde{C}}$. Then there exists $w \in W$ such that $w(p) \in \overline{C}$. Write $w = \tilde{w}x$ with $\tilde{w} \in \widetilde{W}$ and $x \in W_I$. Then $x(p) \in \overline{\widetilde{C}}$ and $\tilde{w}(x(p)) \in \overline{\widetilde{C}}$. Since $\overline{\widetilde{C}}$

is a fundamental domain for $\widetilde{W}$, we get that $\tilde{w}(x(p)) = x(p)$. So $p = x^{-1}(w(p)) \in x^{-1}(\overline{C}) \subseteq \hat{C}$. $\square$

Corollary 4.2. *Let L be a subset of $\widetilde{J}$ such that $\widetilde{W}_L$ is finite. Then there exists a subset K of S and an element d of $X_{I\cap K}^I$ such that W_K is finite and $\widetilde{W}_L = \widetilde{W} \cap {}^d W_K$.*

Proof. Since $\widetilde{W}_L$ is finite, there exists $p \in \overline{\widetilde{C}}$ such that $\widetilde{W}_L = \mathrm{Stab}_{\widetilde{W}}(p)$ (see [4, Chapter V, §3]). By Proposition 4.1, there exists $x \in W_I$ such that $x(p) \in \overline{C}$. Let K be the subset of S such that $\mathrm{Stab}_W(x(p)) = W_K$ (see [4, Chapter V, §3, Proposition 1]). Then W_K is finite and $\widetilde{W}_L = \widetilde{W} \cap {}^{x^{-1}} W_K$. Now, let d be the unique element of minimal length in $x^{-1} W_{I\cap K}$. Then $d \in X_{I\cap K}^I$ and $\widetilde{W}_L = \widetilde{W} \cap {}^d W_K$. $\square$

Corollary 4.3. *Assume that (W, S) is an irreducible affine Weyl group and that $J \neq \varnothing$. Then all the irreducible components of $\widetilde{W}$ are affine.*

Proof. Since (W, S) is affine and irreducible and $I \subsetneq S$, the group W_I is finite. Therefore, $\overline{\widetilde{C}}$ is compact and the result follows. $\square$

Remark - The two previous corollaries could have been shown using the classification and the Table given at the end of this paper.

5. Finite Coxeter groups

In this section, and only in this section, we assume that W is *finite*. We shall relate here the semidirect product decomposition with other constructions which are particular to the finite case: invariants, Solomon algebra. We first start by an easy result:

Proposition 5.1. *If (W, S) is finite and irreducible and if $J \neq \varnothing$, then $|\widetilde{J}| = |S|$.*

Remark - Of course, the above proposition is easily checked using the classification (see the Table at the end of this paper). We shall provide here a general proof. As it is also shown by this table, the proposition is no longer true in general if we do not assume that W is finite.

Proof. We assume without loss of generality that (W, S) is the Coxeter system associated to a based root system (Φ, Π) such that Π is linearly independent. We keep the notation of the proof of Theorem 3.6 (Π_J, $\widetilde{\Pi}$...).

First, $\widetilde{\Pi} \subseteq \Phi^+$. Let $(\lambda_\alpha)_{\alpha\in\widetilde{\Pi}}$ be a family of real numbers such that $\sum_{\alpha\in\widetilde{\Pi}} \lambda_\alpha \alpha = 0$. Let $x = \sum_{\alpha\in\widetilde{\Pi}} |\lambda_\alpha|\, \alpha$. Since $\langle\, \alpha, \beta \,\rangle \leqslant 0$ if α, $\beta \in \widetilde{\Pi}$ (see $(*)$ in the proof of Theorem 3.6) and since $\langle\, ,\, \rangle$ is positive definite, we get that $x = 0$ because

$$\langle\, x, x \,\rangle \leqslant \langle \sum_{\alpha\in\widetilde{\Pi}} \lambda_\alpha \alpha, \sum_{\alpha\in\widetilde{\Pi}} \lambda_\alpha \alpha \rangle = 0.$$

But $\widetilde{\Pi}$ is positively independent, so we get that $\lambda_\alpha = 0$ for all $\alpha \in \widetilde{\Pi}$. Therefore, $\widetilde{\Pi}$ is linearly independent, so $|\widetilde{\Pi}| \leqslant |S|$.

Since $|\widetilde{J}| = |\widetilde{\Pi}|$, it remain to show that $|\widetilde{\Pi}| \geqslant |S|$ or, in other words, that $\widetilde{\Pi}$ generates $\mathscr{E}$. Let $\mathscr{E}'$ be the subspace generated by $\widetilde{\Pi}$. It is W_I-stable by definition of $\widetilde{\Pi}$ and it is $\widetilde{W}$-stable since $\widetilde{W}$ is generated by the (orthogonal) reflections $(s_\alpha)_{\alpha\in\widetilde{\Pi}}$. So $\mathscr{E}'$ is W-stable by Theorem 1.1 (a). Since $\mathscr{E}$ is an irreducible W-module and since $\widetilde{\Pi} \neq \varnothing$, we get that $\mathscr{E}' = \mathscr{E}$, as expected. $\square$

Invariant theory. Keep the notation of the proof of the Proposition 5.1. We view $\mathscr{E}$ as an algebraic variety over $\mathbb{R}$. The group $W/\widetilde{W} \simeq W_I$ acts linearly on the tangent space $\mathscr{T}$ to $\mathscr{E}/\widetilde{W}$ at 0. Since $\widetilde{W}$ is finite and generated by reflections, this tangent space has dimension $\dim \mathscr{E}$ (since $\mathscr{E}/\widetilde{W}$ is smooth). Moreover, by [1, Theorems 3.2 and 3.12, Proposition 3.5], we have:

Proposition 5.2. *The group W_I acts (faithfully) as a reflection group on $\mathscr{T}$: a reflection in W_I acts as a reflection on $\mathscr{T}$.*

REMARK - In [1], the authors have investigated the links between different objects associated to the invariant theory of W and $W/\widetilde{W} \simeq W_I$: degrees, hyperplane arrangements, fake degrees, regular elements...

Solomon descent algebra. If $K \subseteq S$ and $L \subseteq \widetilde{J}$, we set

$$x_K = \sum_{w\in X_K} w \in \mathbb{Q}W, \qquad \tilde{x}_L = \sum_{w\in \widetilde{X}_L} w \in \mathbb{Q}\widetilde{W},$$

and

$$x_L = \sum_{w\in X_L} w \in \mathbb{Q}W.$$

The *Solomon descent algebra* $\Sigma(W)$ of W is defined by

$$\Sigma(W) = \mathop{\oplus}_{K\subseteq S} \mathbb{Q}x_K$$

(see [9]). It turns out that it is a subalgebra of the group algebra $\mathbb{Q}W$. Similarly, we set

$$\Sigma(\widetilde{W}) = \mathop{\oplus}_{L\subseteq \widetilde{J}} \mathbb{Q}\tilde{x}_L.$$

We then define a $\mathbb{Q}$-linear map

$$\widetilde{\mathrm{Res}} : \Sigma(W) \longrightarrow \Sigma(\widetilde{W})$$

by

$$\widetilde{\mathrm{Res}}(x_K) = \sum_{d\in X_{I\cap K}^I} \tilde{x}_{\widetilde{J}\cap {}^d W_K} \qquad (= \sum_{d\in X_{I\cap K}^I} \tilde{x}_{{}^d K^+})$$

for all $K \subseteq S$.

Proposition 5.3. *The map $\widetilde{\mathrm{Res}} : \Sigma(W) \to \Sigma(\widetilde{W})$ is a morphism of $\mathbb{Q}$-algebras.*

Proof. Let $z = \sum_{w \in W_I} w$. We shall first show that, for $x \in \Sigma(W)$,

$$z\,\widetilde{\mathrm{Res}}(x) = xz. \tag{5.10}$$

For this, we may assume that $x = x_K$ for some $K \subseteq S$. Let $z' = \sum_{w \in W_{I\cap K}} w$. Since

$$z = \sum_{d \in X^I_{I\cap K}} z' d^{-1},$$

we get

$$x_K z = (x_K z') \cdot \sum_{d \in X^I_{I\cap K}} d^{-1}.$$

But $W_{I\cap K}$ is the set of elements $w \in W_K$ of minimal length in $w\widetilde{W}_{K^+}$. So it follows from Lemma 1.2 (c) that the map

$$\begin{array}{ccc} X_K \times W_{I\cap K} & \longrightarrow & X_{K^+} \\ (x, w) & \longmapsto & xw \end{array}$$

is bijective. So $x_K z' = x_{K^+} = z\tilde{x}_{K^+}$. Therefore,

$$x_K z = z \sum_{d \in X^I_{I\cap K}} \tilde{x}_{\widetilde{J}\cap W_K} d^{-1}.$$

But $zd = z$ for all $d \in X^I_{I\cap K}$, so

$$x_K z = z \sum_{d \in X^I_{I\cap K}} d\tilde{x}_{\widetilde{J}\cap W_K} d^{-1}.$$

Since W_I acts on the pair $(\widetilde{W}, \widetilde{J})$, we have $d\widetilde{X}_{\widetilde{J}\cap W_K} d^{-1} = \widetilde{X}_{\widetilde{J}\cap {}^d W_K}$, so 5.10 follows.

Since the map $\mathbb{Z}\widetilde{W} \to \mathbb{Z}W$, $u \mapsto zu$ is injective, we get immediately from 5.10 that $\widetilde{\mathrm{Res}}$ is a morphism of rings. $\square$

Note that the group W_I acts by conjugation on the descent algebra of $\widetilde{W}$.

Corollary 5.4. *The image of* $\widetilde{\mathrm{Res}}$ *is* $\Sigma(\widetilde{W})^{W_I}$.

Proof. Let $x \in \Sigma(W)$ and $w \in W_I$. Then, by 5.10,

$$z \cdot {}^w\widetilde{\mathrm{Res}}(x) = zw\,\widetilde{\mathrm{Res}}(x) w^{-1} = z\,\widetilde{\mathrm{Res}}(x) w^{-1} = xzw^{-1} = xz = z\,\widetilde{\mathrm{Res}}\,x,$$

so $\widetilde{\mathrm{Res}}(x) = {}^w\widetilde{\mathrm{Res}}(x)$. This shows that the image of $\widetilde{\mathrm{Res}}$ is contained in $\Sigma(\widetilde{W})^{W_I}$.

Conversely, we need to show that, for all $L \subseteq \widetilde{J}$, the element $u = \sum_{w \in W_I} \tilde{x}_{wLw^{-1}}$ is in the image of $\widetilde{\mathrm{Res}}$. But, by Corollary 4.2, there exists $K \subseteq S$ and $d \in X^I_{I\cap K}$ such that $\widetilde{W}_L = \widetilde{W} \cap {}^d W_K$. Then $L = dK^+d^{-1}$ and

$$\widetilde{\mathrm{Res}}(x_K) = \sum_{x \in X^I_{I\cap K}} \tilde{x}_{xK^+} = \frac{1}{|W_{I\cap K}|} u,$$

as desired. $\square$

The Solomon descent algebra $\Sigma(W)$ is endowed with a morphism of $\mathbb{Q}$-algebras $\theta : \Sigma(W) \longrightarrow \mathbb{Q}\operatorname{Irr} W$, where $\mathbb{Q}\operatorname{Irr} W$ denotes the algebra of $\mathbb{Q}$-linear combinations of irreducible characters of W (with usual product). The map θ is defined by

$$\theta(x_K) = \operatorname{Ind}_{W_K}^{W} 1_K$$

for all $K \subseteq S$ (here, 1_K denotes the trivial character of W_K). Similarly, we have a morphism of $\mathbb{Q}$-algebras $\tilde{\theta} : \Sigma(\widetilde{W}) \to \mathbb{Q}\operatorname{Irr}\widetilde{W}$ and the Mackey formula shows immediately that the diagram

(5.11)
$$\begin{tikzcd}
\Sigma(W) \arrow[r, "\theta"] \arrow[d, "\widetilde{\operatorname{Res}}"'] & \mathbb{Q}\operatorname{Irr} W \arrow[d, "\operatorname{Res}_{\widetilde{W}}^{W}"] \\
\Sigma(\widetilde{W}) \arrow[r, "\tilde{\theta}"] & \mathbb{Q}\operatorname{Irr}\widetilde{W}
\end{tikzcd}$$

is commutative.

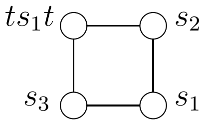
REMARK - In [3, §5.2], the authors have defined the map $\widetilde{\operatorname{Res}}$ whenever W is of type B_n and $\widetilde{W}$ is of type D_n (it was denoted by Res_n). In this particular case, Proposition 5.3, Corollary 5.4 and the commutativity of the diagram 5.11 have been shown in [3, Proposition 5.9].

6. EXAMPLES

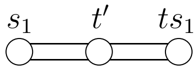
We shall describe in detail some examples of (internal) semidirect product decompositions of Coxeter systems (W, S). If Δ is a Coxeter graph, we shall denote by $W(\Delta)$ the associated Coxeter group. In the following table, we have drawn the diagram of $(W, S, \boldsymbol{I})$ by marking with *black* nodes the elements of I. The elements of $\widetilde{J}$ and their reduced expressions have been obtained using the bijection (1.4) and Proposition 1.6 (b). The Coxeter graph of $(\widetilde{W}, \widetilde{J})$ is obtained from Theorem 1.8, and the action of the Coxeter generators I of W_I by diagram automorphisms of the Coxeter graph of $(\widetilde{W}, \widetilde{J})$ may be determined using Proposition 1.6 (a).

The table contains all possible triples (W, S, I) where W is a finite Coxeter group or an affine Weyl group and is irreducible and I is a proper non-empty subset of S. (For compactness, we include $\widetilde{A}_1$ as $I_2(\infty)$). In degenerate cases, that is, for small values of $|S|$, the diagram for $(\widetilde{W}, \widetilde{J})$ given in the table is not correct, but the semidirect product decomposition is still correct (see the marks (1), (2) and (3) in the table). Here are some detailed explanations:

(1) If W is of type $\widetilde{B}_3$, then, since $D_3 = A_3$, we have $\widetilde{D}_3 = \widetilde{A}_3$. So the correct Coxeter graph of $(\widetilde{W}, \widetilde{J})$ is a square of this form

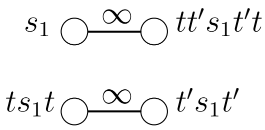


(2) If W is of type $\widetilde{C}_2$, then, since $B_2 = C_2$, we have $\widetilde{B}_2 = \widetilde{C}_2$. So the correct Coxeter graph of $(\widetilde{W}, \widetilde{J})$ is of the following form

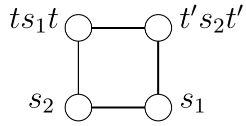


(3) For the diagram marked (3) in the table, there are two values of n for which the graph degenerates: if $n = 2$, then $D_2 = A_1 \times A_1$ (this is a standard convention) and so $\widetilde{D}_2 = \widetilde{A}_1 \times \widetilde{A}_1$ and, if $n = 3$, then $D_3 = A_3$ so again $\widetilde{D}_3 = \widetilde{A}_3$ is a square. We obtain the following diagrams:

$\widetilde{D}_2$



$\widetilde{D}_3$



We next explain the notation t_i and t'_i in the Coxeter graphs marked (a), (b), (c) and (d) in the table.

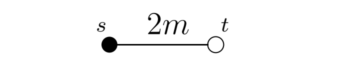
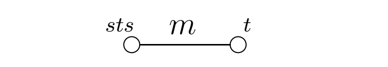
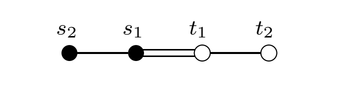
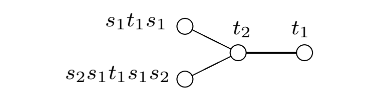
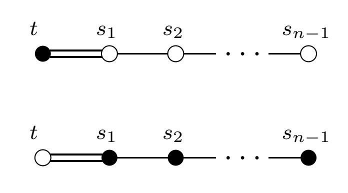
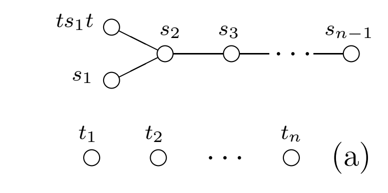
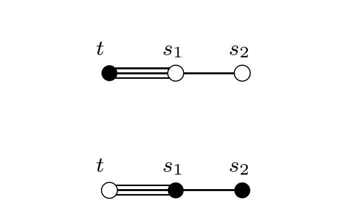
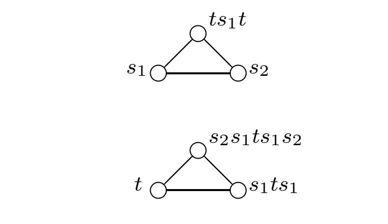
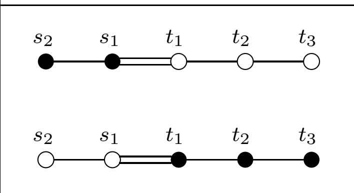
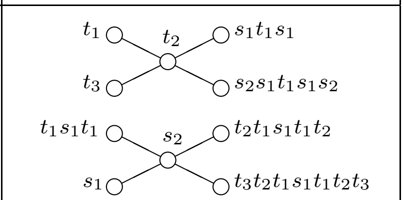
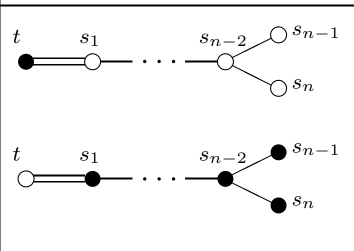
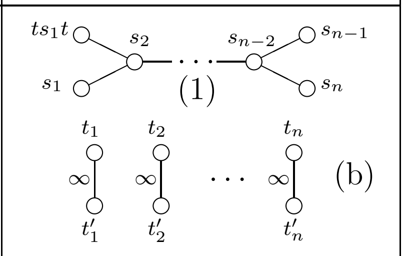
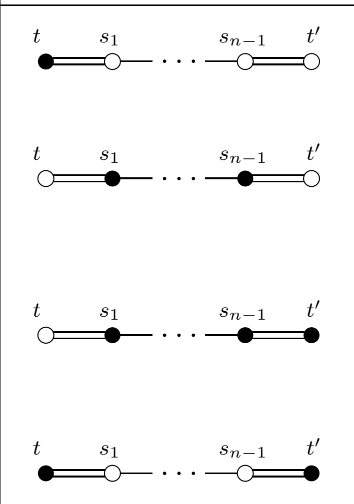
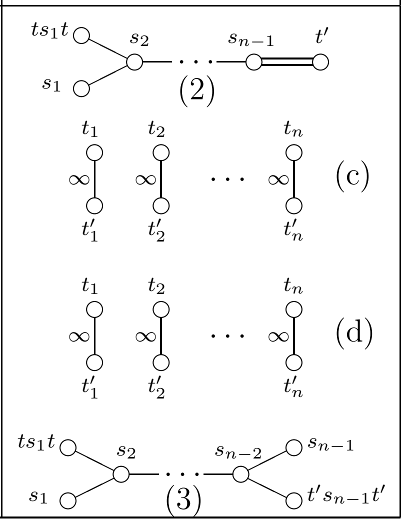
(a) Here, $t_1 = t$ and $t_{i+1} = s_it_is_i$ $(1 \leqslant i \leqslant n-1)$.
(b) Here, $t_1 = t$ and $t_{i+1} = s_it_is_i$ $(1 \leqslant i \leqslant n-1)$, $t'_n = s_nt_{n-1}s_n$ and $t'_i = s_it'_{i+1}s_i$ $(1 \leqslant i \leqslant n-1)$.
(c) Here, $t_1 = t$ and $t_{i+1} = s_it_is_i$ $(1 \leqslant i \leqslant n-1)$, $t'_n = t'$ and $t'_i = s_it'_{i+1}s_i$ $(1 \leqslant i \leqslant n-1)$.
(d) Here, $t_1 = t$ and $t_{i+1} = s_it_is_i$ $(1 \leqslant i \leqslant n-1)$, $t'_n = t't_nt'$ and $t'_i = s_it'_{i+1}s_i$ $(1 \leqslant i \leqslant n-1)$.

Finally, it remains to describe the W_I-action by automorphisms of $(\widetilde{W}, \widetilde{J})$. This may be done by describing the automorphism of the Coxeter graph given by the simple reflections I of W_I. Each $s \in I$ acts by conjugation on the vertex set $\widetilde{J}$ of the Coxeter graph, and in most cases the action is clear by inspection of the graph. It may be specified by giving the induced permutation of the vertex set $\widetilde{J}$ of the Coxeter graph. For example, in type $\widetilde{G}_2$ with $I = \{s_1, s_2\}$, the action is given by $s_1 \mapsto (t, s_1ts_1)$ and $s_2 \mapsto (s_1ts_1, s_2s_1ts_1s_2)$ where the image permutations are written in disjoint cycle notation. We will not explicitly list the action in the cases in which it is obvious by inspection.

The four graphs in the table (or amongst the degenerate graphs discussed above) for which the action is perhaps not obvious by inspection are again those designated (a), (b), (c) and (d). For these, the actions of W_I are as follows:

(a) Here, $s_i \mapsto (t_i, t_{i+1})$ for $1 \leqslant i \leq n-1$.
(b) Here, $s_i \mapsto (t_i, t_{i+1})(t'_i, t'_{i+1})$ for $1 \leqslant i \leqslant n-1$, and $s_n \mapsto (t_{n-1}, t'_n)(t'_{n-1}, t_n)$.
(c) Here, $s_i \mapsto (t_i, t_{i+1})(t'_i, t'_{i+1})$ for $1 \leqslant i \leqslant n-1$.
(d) Here, $s_i \mapsto (t_i, t_{i+1})(t'_i, t'_{i+1})$ for $1 \leqslant i \leqslant n-1$ and $s_n \mapsto (t_n, t'_n)$.

The resulting permutation representation of W_I is in each case (a)–(d) isomorphic in an obvious way to a standard permutation representation of the classical Weyl group W_I as a group of permutations or signed permutations.

<table>
<tr><th>Type</th><th>Graph of $(W, S, \boldsymbol{I})$</th><th>Decomposition</th><th>Graph of $(\widetilde{W}, \widetilde{J})$</th></tr>
<tr><td>$I_2(2m)$</td><td></td><td>$(\mathbb{Z}/2\mathbb{Z}) \ltimes W(I_2(m))$</td><td></td></tr>
<tr><td>F_4</td><td></td><td>$\mathfrak{S}_3 \ltimes W(D_4)$</td><td></td></tr>
<tr><td>B_n
$(n \geqslant 2)$</td><td></td><td>$(\mathbb{Z}/2\mathbb{Z}) \ltimes W(D_n)$
$\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$</td><td></td></tr>
<tr><td>$\widetilde{G}_2$</td><td></td><td>$(\mathbb{Z}/2\mathbb{Z}) \ltimes W(\widetilde{A}_2)$
$\mathfrak{S}_3 \ltimes W(\widetilde{A}_2)$</td><td></td></tr>
<tr><td>$\widetilde{F}_4$</td><td></td><td>$\mathfrak{S}_3 \ltimes W(\widetilde{D}_4)$
$\mathfrak{S}_4 \ltimes W(\widetilde{D}_4)$</td><td></td></tr>
<tr><td>$\widetilde{B}_n$
$(n \geqslant 3)$</td><td></td><td>$(\mathbb{Z}/2\mathbb{Z}) \ltimes W(\widetilde{D}_n)$
$W(D_n) \ltimes \big(W(\widetilde{A}_1)\big)^n$</td><td></td></tr>
<tr><td>$\widetilde{C}_n$
$(n \geqslant 2)$</td><td></td><td>$(\mathbb{Z}/2\mathbb{Z}) \ltimes W(\widetilde{B}_n)$
$\mathfrak{S}_n \ltimes \big(W(\widetilde{A}_1)\big)^n$
$W(B_n) \ltimes \big(W(\widetilde{A}_1)\big)^n$
$(\mathfrak{S}_2 \times \mathfrak{S}_2) \ltimes W(\widetilde{D}_n)$</td><td></td></tr>
</table>

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